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SHORT TIME BEHAVIOR OF THE HEAT KERNEL AND ITS LOGARITHMIC DERIVATIVES

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Abstract

Let M be a compact, connected Riemannian manifold, and let $p_t(x,y)$ denote the fundamental solution to Cauchy initial value problem for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$, where Δ is the Levi–Civita Laplacian. The purpose of this note is to study the asymptotic behavior of derivatives of $\log p_t(\cdot, y)$ at x as $t \searrow 0$. In particular, we show that a dramatic change takes place when x moves inside the cut-locus of y.

0. Introduction

Let M be a compact, connected, d-dimensional Riemannian manifold, denote by $\mathcal{O}(\mathcal{M})$ with fiber map $\pi : \mathcal{O}(\mathcal{M}) \longrightarrow M$ the associated bundle of orthonormal frames $\mathfrak{e}$, and use the Levi–Civita connection to determine the horizontal subspace $H_{\mathfrak{e}}(\mathcal{O}(\mathcal{M}))$ at each $\mathfrak{f} \in \mathcal{O}(\mathcal{M})$. Next, given $\mathbf{v} \in \mathbb{R}^d$, let $\mathfrak{E}(\mathbf{v})$ be the *basic vector field* on $\mathcal{O}(\mathcal{M})$ determined by properties that

$$\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} \in H_{\mathfrak{e}}(\mathcal{O}(\mathcal{M})) \quad \text{and} \quad d\pi\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} = \mathfrak{e}\mathbf{v} \quad \text{for all } \mathfrak{e} \in \mathcal{O}(\mathcal{M}).$$

(Here, and whenever convenient, we think of $\mathfrak{e}$ as an isometry from $\mathbb{R}^d$ onto $T_{\pi(\mathfrak{e})}(M)$.) In particular, if $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard orthonormal basis in $\mathbb{R}^d$, then we set $\mathfrak{E}_k(\mathfrak{e}) = \mathfrak{E}(\mathbf{e}_k)_{\mathfrak{e}}$. If, for $\mathcal{O} \in O(d)$ (the orthogonal group on $\mathbb{R}^d$) $R_{\mathcal{O}} : \mathcal{O}(\mathcal{M}) \longrightarrow \mathcal{O}(\mathcal{M})$ is defined so that

$$R_{\mathcal{O}}\mathfrak{e}\,\mathbf{v} = \mathfrak{e}\,\mathcal{O}\mathbf{v}, \quad \mathfrak{e} \in \mathcal{O}(\mathcal{M}) \text{ and } \mathbf{v} \in \mathbb{R}^d,$$

then it easy to check that

$$dR_{\mathcal{O}}\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} = \mathfrak{E}(\mathcal{O}^{\top}\mathbf{v})_{R_{\mathcal{O}}\mathfrak{e}}, \quad \mathfrak{e} \in \mathcal{O}(\mathcal{M}) \text{ and } \mathbf{v} \in \mathbb{R}^d. \tag{0.1}$$

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Given a smooth function F on $\mathcal{O}(\mathcal{M})$, we define $\nabla F : \mathcal{O}(\mathcal{M}) \longrightarrow \mathbb{R}^d$, $\operatorname{Hess}(F) : \mathcal{O}(\mathcal{M}) \longrightarrow \operatorname{Hom}(\mathbb{R}^d;\mathbb{R}^d)$, and $\Delta F : \mathcal{O}(\mathcal{M}) \longrightarrow \mathbb{R}$ by

$$\nabla F = \sum_1^d \mathfrak{E}_k F\, \mathbf{e}_k, \quad \operatorname{Hess}(F) = \big((\mathfrak{E}_k \circ \mathfrak{E}_\ell F)\big)_{1\le k,\ell\le d} \tag{0.2}$$
$$\text{and} \quad \Delta F = \sum_1^d \mathfrak{E}_k^2 F.$$

In particular, when f is a smooth function on M, we set

$$\nabla f \equiv \nabla(f\circ\pi), \quad \operatorname{Hess}(f) \equiv \operatorname{Hess}(f\circ\pi), \quad \text{and} \quad \Delta f \equiv \Delta(f\circ\pi).$$

Starting from (0.1), it is an easy matter to check that

$$(\nabla f)\circ R_{\mathcal{O}} = \mathcal{O}^\top \nabla f, \quad \big(\operatorname{Hess}(f)\big)\circ R_{\mathcal{O}} = \mathcal{O}^\top \operatorname{Hess}(f)\,\mathcal{O},$$
$$\text{and} \quad (\Delta f)\circ R_{\mathcal{O}} = \Delta f.$$

Hence, $|\nabla f|$, $\|\operatorname{Hess}(f)\|_{\text{H.S.}}$ (the Hilbert–Schmidt norm), and Δf are all well-defined on M. In fact, Δf is precisely the action of the Levi–Civita Laplacian on f.

Now consider the Cauchy initial value for the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\Delta u, \quad t\in(0,\infty) \quad \text{with} \quad \lim_{t\searrow 0} u(t,x) = f(x), \quad x\in M.$$

By standard elliptic regularity theory, one knows that there is a unique, smooth function $(t,x,y)\in(0,\infty)\times M\times M \longmapsto p_t(x,y)\in(0,\infty)$ such that

$$u(t,x) = \int_M f(y)\,p_t(x,y)\,\lambda_M(dy), \quad (t,x)\in(0,\infty)\times M,\ f\in C(M;\mathbb{R}),$$

where λ_M denotes the normalized Riemann measure on M. Moreover, because Δ is essentially self-adjoint in $L^2(\lambda_M)$, $p_t(x,y) = p_t(y,x)$.

By any one of a number of different procedures, one can obtain Varadhan's result:

$$\lim_{T\searrow 0} T\log p_T(x,y) = -\frac{\operatorname{dist}(x,y)^2}{2}, \quad x,y\in M. \tag{0.3}$$

In fact, the limit in (0.3) is taken uniformly with respect to $(x,y)\in M^2$. The probabilistic intuition behind Varadhan's result comes from the Feynman-type path integral representation

$$p_T(x,y) = C(T) \int_{p(0)=x\,\&\,p(1)=y} \exp\left(-\frac{1}{2T}\int_0^1 |\dot p(t)|^2\,dt\right)\mathfrak{D}p, \tag{0.4}$$

where the right-hand side is supposed to convey the idea that one is integrating over all paths $p \in C([0,1];M)$ which run from x to y, and one is weighting paths in a Gibbsian manner according to their energy. Considering what utter non-sense (0.4), as it stands, is, results like Varadhan's are surprising. But, experience has taught us that, ridiculous as it appears, (0.4) is, nonetheless, unreasonably correct; a conclusion for which the present article can be viewed as further corroboration. In fact, basing our reasoning on the intuition coming from (0.4), our aim here is to examine what happens, as $T \searrow 0$, to derivatives of $\log p_T(\,\cdot\,,y)$. Obviously, as soon as one starts taking derivatives, one should expect the behavior outside the cut-locus to be different from that inside the cut-locus, where the distance function is no longer smooth. In terms of (0.4), what one suspects is that the problems will arise from a breakdown of the Laplace asymptotic method due to the degeneracy of the minimization problem

$$\min\left\{\int_0^1 |\dot{p}(t)|^2\,dt : p(0) = x \text{ and } p(1) = y\right\}$$

when x lies inside the cut-locus of y. As the development which we give below makes manifest, this is precisely what happens.

In Section 1, we develop explicit formulae (cf. (1.8) and (1.9)), in terms of integrals with respect to Wiener's measure, for the first two spatial derivatives of $\log p_T(\,\cdot\,,y)$. Consideration of the cut-locus does not affect the validity of these formulae, but its potential rôle is already evident in the expression (1.9) for the logarithmic Hessian. Namely, that expression segregates naturally into terms of order T^{-1} and terms of order T^{-2}. Terms of order T^{-1} are what one should expect on the basis of (0.3). In particular, if one hopes to exchange two derivatives with the limit in (0.3), then one must show that the terms of order T^{-2} in (1.9) disappear in the limit. Remarkably, these terms of order T^{-2} can be recognized as a *variance*. Hence, if, in a sufficiently strong sense, the Wiener integral is concentrating (as $T \searrow 0$) along a single path, then this term should tend to 0 because the random variable of which it is the variance is becoming constant. On the other hand, if there is more than one path to which the Wiener integral is giving mass in the limit, then this variance should remain positive and the terms of order T^{-2} will become the dominant ones. Thus, the existence of more than one minimizing geodesic (or even of a non-trivial Jacobi field) has the potential to radically change the behavior of the logarithmic Hessian.

The asymptotic analysis of (1.8) and (1.9) is carried out in Section 2. What is involved is an application of the theory of large deviations, as developed in [3]. (Closely related applications are given in [4].) What we show (cf. Corollary 2.29) is that as long as x stays outside the cut-locus of y (0.3) holds, even after taking derivatives up to the second order. On the other hand, if x lies at the cut-locus of y, then (cf. Theorem 2.35) (0.3) may break down, even after taking only one derivative. In fact, under additional technical conditions, we show that the second derivatives of $\log p_T(\,\cdot\,,y)$ will be of order T^{-2} at the cut-locus of y.

1. Logarithmic derivatives of the heat flow semigroup

In this section, we re-formulate some results from [5] in a way which makes them more amenable to the theory of large deviations as it was developed in [3].

Let $\mathfrak{W}$ be the separable Banach space (with respect to the uniform convergence) of continuous paths $\mathbf{w} : [0,1] \longrightarrow \mathbb{R}^d$ satisfying $\mathbf{w}(0) = \mathbf{0}$, and use $\mathcal{B}_{\mathfrak{W}}$ to denote the Borel field over $\mathfrak{W}$. In addition, for each $t \in [0,1]$, $\mathcal{B}_t$ will denote the σ-algebra generated by $\mathbf{w} \in \mathfrak{W} \longmapsto \mathbf{w}(\tau) \in \mathbb{R}^d$ as τ varies over $[0,t]$. Finally, we will use μ to denote the standard Wiener measure on $(\mathfrak{W}, \mathcal{B}_{\mathfrak{W}})$, and, for each $T \in (0,1]$ we take μ_T to be the distribution of $\mathbf{w} \in \mathfrak{W} \longmapsto \sqrt{T}\,\mathbf{w} \in \mathfrak{W}$ under μ.

Next, given a frame $\mathfrak{e} \in \mathcal{O}(\mathcal{M})$ and $T \in (0,1]$, define $\mathfrak{F}_{\mathfrak{e}} : [0,\infty) \times \mathfrak{W} \longrightarrow \mathcal{O}(\mathcal{M})$ to be the μ_T-almost surely unique, progressively measurable (relative to $\{\mathcal{B}_t : t \geq 0\}$) solution to the Stratonovich stochastic differential equation[1]

$$d\mathfrak{F}_{\mathfrak{e}}(t,\mathbf{w}) = \sum_{k=1}^{d} \mathfrak{E}_k\big(\mathfrak{F}_{\mathfrak{e}}(t,\mathbf{w})\big) \circ d\mathbf{w}(t)_k \quad \text{with } \mathfrak{F}_{\mathfrak{e}}(0,\mathbf{w}) = \mathfrak{e}.$$

As an easy application of Itô's formula and (0.2), one sees that, for any $T \in [0,1]$ and $f \in C(M;\mathbb{R})$,

$$\text{(1.1)} \quad \mathbb{E}^{\mu_T}\Big[(f\circ\pi)\big(\mathfrak{F}_{\mathfrak{e}}(1)\big)\Big] = \big[P_T f\big]\big(\pi(\mathfrak{e})\big) \equiv \int_M f(y)\, p_T\big(\pi(\mathfrak{e}),y\big)\, \lambda_M(dy).$$

We will next use the procedure developed in [2] and [5] to pass from (1.1) to representations of derivatives of $\log p_T(\,\cdot\,,y)$. Unfortunately, this will require some additional notation.

The *solder form* $\omega : T\big(\mathcal{O}(\mathcal{M})\big) \longrightarrow \mathbb{R}^d$ is the 1-form defined so that, for each $\mathfrak{e} \in \mathcal{O}(\mathcal{M})$ and $X_{\mathfrak{e}} \in T_{\mathfrak{e}}\big(\mathcal{O}(\mathcal{M})\big)$, $d\pi X_{\mathfrak{e}} = \mathfrak{e}\omega(X_{\mathfrak{e}})$. Thus, the vertical subspace at $\mathfrak{e}$ is precisely the null space of $\omega \restriction T_{\mathfrak{e}}\big(\mathcal{O}(\mathcal{M})\big)$. Next, let $\mathfrak{o}(d)$

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[1] Obviously, the full definition of $\mathbf{w} \in \mathfrak{W} \longmapsto \mathfrak{F}_{\mathfrak{e}}(\,\cdot\,,\mathbf{w}) \in C\big([0,1];M\big)$ really depends on T, but we have chosen to suppress this dependence in the interest of simplifying notation.

stand for the Lie algebra of skew symmetric $d \times d$-matrices, remember that $\mathfrak{o}(d)$ can be identified with the Lie algebra of left-invariant vector fields on $O(d)$, and let λ be the map of $\mathfrak{o}(d)$ into the $T(\mathcal{O}(\mathcal{M}))$ given by

$$\lambda(A)_{\mathfrak{e}} = \frac{d}{dt} R_{e^{tA}}\mathfrak{e}\Big|_{t=0}, \quad A \in \mathfrak{o}(d) \text{ and } \mathfrak{e} \in \mathcal{O}(\mathcal{M}).$$

Clearly, $A \in \mathfrak{o}(d) \longmapsto \lambda(A)_{\mathfrak{e}} \in T_{\mathfrak{e}}(\mathcal{O}(\mathcal{M}))$ provides an isomorphism between $\mathfrak{o}(d)$ and the vertical subspace at $\mathfrak{e}$. Thus, we can define the *connection 1-form* $\phi : T(\mathcal{O}(\mathcal{M})) \longrightarrow \mathfrak{o}(d)$ so that, for each $\mathfrak{e} \in \mathcal{O}(\mathcal{M})$ and $X_{\mathfrak{e}} \in T_{\mathfrak{e}}(\mathcal{O}(\mathcal{M}))$,

$$X_{\mathfrak{e}} - \lambda(\phi(X_{\mathfrak{e}})) = \sum_{k=1}^{d} \omega(X_{\mathfrak{e}})_k \mathfrak{E}_k(\mathfrak{e}) \quad \text{is the horizontal part of } X_{\mathfrak{e}}.$$

Equivalently, $\lambda(\phi(X_{\mathfrak{e}}))$ is the vertical part of $X_{\mathfrak{e}}$. Finally, the *Riemann curvature 2-form* $\Phi : T(\mathcal{O}(\mathcal{M}))^2 \longrightarrow \mathfrak{o}(d)$ is the horizontal part of the exterior derivative $d\phi$ of ϕ. We set

$$\text{(1.2)} \qquad \Phi(\mathbf{v}, \mathbf{v}')_{\mathfrak{e}} = \Phi(\mathfrak{E}(\mathbf{v})_{\mathfrak{e}}, \mathfrak{E}(\mathbf{v}')_{\mathfrak{e}}), \quad \mathfrak{e} \in \mathcal{O}(\mathcal{M}) \text{ and } \mathbf{v}, \mathbf{v}' \in \mathbb{R}^d,$$

define the Ricci curvature matrix $\mathrm{Ric} : \mathcal{O}(\mathcal{M}) \longrightarrow \mathrm{Hom}(\mathbb{R}^d; \mathbb{R}^d)$ by

$$\Big(\mathbf{v}, \mathrm{Ric}(\mathfrak{e})\mathbf{v}'\Big)_{\mathbb{R}^d} = \sum_{k=1}^{d} \Big(\Phi(\mathbf{v}, \mathbf{e}_k)_{\mathfrak{e}} \mathbf{e}_k, \mathbf{v}'\Big)_{\mathbb{R}^d}, \quad \mathbf{v}, \mathbf{v}' \in \mathbb{R}^d,$$

and, for each $T \in (0, 1]$, determine the progressively measurable map $\mathbf{A}_{\mathfrak{e},T} : [0,1] \times \mathfrak{W} \longrightarrow \mathrm{Hom}(\mathbb{R}^d; \mathbb{R}^d)$ by

$$\text{(1.3)} \qquad \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) + \frac{T}{2} \int_0^t \mathrm{Ric}(\mathfrak{F}_{\mathfrak{e}}(\tau, \mathbf{w})) \mathbf{A}_{\mathfrak{e},T}(\tau, \mathbf{w})\, d\tau = \mathbf{I}.$$

The following equation is a minor generalization of (2.2) in [5], when one takes into account the use of μ_T in place of μ:

(1.4)

$$\begin{aligned} T \quad & \Big[\mathfrak{E}(\mathbf{v}) \log P_T f \circ \pi\Big](\mathfrak{e}) \\ & = \big(P_T f(x)\big)^{-1} \mathbb{E}^{\mu_T} \left[\int_0^1 \Big(\mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d} f \circ \pi(\mathfrak{F}_{\mathfrak{e}}(1, \mathbf{w})) \right] \end{aligned}$$

for any $f \in C(M; (0, \infty))$ and $\boldsymbol{\eta} \in C^1([0,1]; \mathbb{R}^d)$ with $\boldsymbol{\eta}(0) = \mathbf{0}$ and $\boldsymbol{\eta}(1) = \mathbf{v} \in \mathbb{R}^d$.

We next want to make the analogous translation of (2.12) in [5]. For this purpose, let $\boldsymbol{\eta} \in \mathbf{H}$ be given, and define the progressively measurable map $\phi_{\mathfrak{e},T,\boldsymbol{\eta}} : [0,1] \times \mathfrak{W} \longmapsto \mathfrak{o}(d)$ so that

$$
\begin{aligned}
&\Big(\boldsymbol{\xi}', \phi_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\boldsymbol{\xi}\Big)_{\mathbb{R}^d} \\
&\quad = \int_0^t \Big(\Phi(\boldsymbol{\xi},\boldsymbol{\xi}')_{\mathfrak{F}_{\mathfrak{e}}(\tau,\mathbf{w})}\mathbf{A}_{\mathfrak{e},T}(\tau,\mathbf{w})\big(\boldsymbol{\eta}(1)-\boldsymbol{\eta}(\tau)\big), \circ d\mathbf{w}(\tau)\Big)_{\mathbb{R}^d}, \\
&\qquad\qquad \boldsymbol{\xi},\boldsymbol{\xi}' \in \mathbb{R}^d.
\end{aligned} \tag{1.5}
$$

Then, for $f \in C\big(M;(0,\infty)\big)$ and $\boldsymbol{\eta} \in C^2\big([0,1];\mathbb{R}^d\big)$ with $\boldsymbol{\eta}(0) = \mathbf{0}$ and $\boldsymbol{\eta}(1) = \mathbf{v}$:

$$
\begin{aligned}
&T\Big(\mathbf{v}, \big[\mathrm{Hess}\log P_T f\circ\pi\big](\mathfrak{e})\mathbf{v}\Big)_{\mathbb{R}^d} \\
=&T\big[\mathfrak{E}(\mathbf{v})^2 \log P_T f\circ\pi\big](\mathfrak{e}) \\
=&-\big(P_T f(x)\big)^{-1}\Bigg(\mathbb{E}^{\mu_T}\Bigg[\Bigg(\int_0^1 \big|\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t)\big|^2\,dt \\
&\qquad -\int_0^1\Big(\phi_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), \circ d\mathbf{w}(t)\Big)_{\mathbb{R}^d} \\
&\qquad + T\int_0^1 \big(\mathbf{w}(1)-\mathbf{w}(t), \mathbf{R}_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\big)_{\mathbb{R}^d}\,dt\Bigg) f\circ\pi\big(\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})\big)\Bigg]\Bigg) \\
&+\frac{1}{T}\Bigg(\big(P_T f(x)\big)^{-1}\mathbb{E}^{\mu_T}\Bigg[\Bigg(\int_0^1\Big(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d}\Bigg)^2 f\circ\pi\big(\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})\big)\Bigg] \\
&\qquad -\big(P_T f(x)\big)^{-2}\mathbb{E}^{\mu_T}\Bigg[\int_0^1\Big(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d} f\circ\pi\big(\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})\big)\Bigg]^2\Bigg),
\end{aligned} \tag{1.6}
$$

where $\mathbf{R}_{\mathfrak{e},T,\boldsymbol{\eta}} : [0,1]\times\mathfrak{W} \longrightarrow \mathbb{R}^d$ is a progressively measurable function which satisfies

$$
\big|\mathbf{R}_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\big| \le C\|\boldsymbol{\eta}\|_{C^2([0,1];\mathbb{R}^d)} \tag{1.7}
$$

for some $C < \infty$.

Clearly (1.4) and (1.6) can be interpreted in terms of conditional expectations. In fact, because all the functionals involved are smooth in the sense of the Sobolev calculus on Wiener space and, in addition, the map $\mathfrak{F}_{\mathfrak{e}}(1,\cdot)$ is non-degenerate, one can, for each $y \in M$, take f in both (1.4) and (1.6) to be the Dirac delta function δ_y (relative to the Riemannian volume measure on M), in which case (1.4) becomes

$$
\begin{aligned}
&T\big[\mathfrak{E}(\mathbf{v})\log p_T(\,\cdot\,,y)\big](\mathfrak{e}) \\
&\quad = \mathbb{E}^{\mu_T}\Bigg[\int_0^1\Big(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d}\,\Bigg|\,\pi\circ\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\Bigg]
\end{aligned} \tag{1.8}
$$

and (1.6) becomes

$$
\begin{aligned}
T&\Big(\mathbf{v}, [\operatorname{Hess}\log p_T(\,\cdot\,,y)](\mathfrak{e})\mathbf{v}\Big)_{\mathbb{R}^d}\\
&=T\big[\mathfrak{E}(\mathbf{v})^2 \log p_T(\,\cdot\,,y)\big](\mathfrak{e})\\
&=-\,\mathbb{E}^{\mu_T}\Bigg[\int_0^1 \big|\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t)\big|^2\,dt\\
&\qquad\qquad - \int_0^1 \Big(\phi_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), \circ d\mathbf{w}(t)\Big)_{\mathbb{R}^d}\\
&\qquad\qquad + T\int_0^1 \big(\mathbf{w}(1)-\mathbf{w}(t), \mathbf{R}_{\mathfrak{e},T,\boldsymbol{\eta}}(t,\mathbf{w})\big)_{\mathbb{R}^d}\,dt\,\Bigg|\,\pi\circ\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})=y\Bigg]\\
&\quad+\frac{1}{T}\Bigg(\mathbb{E}^{\mu_T}\Bigg[\bigg(\int_0^1\Big(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d}\bigg)^2\,\Bigg|\,\pi\circ\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})=y\Bigg]\\
&\qquad\qquad - \mathbb{E}^{\mu_T}\Bigg[\int_0^1\Big(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}(t), d\mathbf{w}(t)\Big)_{\mathbb{R}^d}\,\Bigg|\,\pi\circ\mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w})=y\Bigg]^2\Bigg),
\end{aligned}
\tag{1.9}
$$

Remark 1.10. Note that, in (1.8) and (1.9), there are no almost everywhere statements accompanying the conditional expectations. This is because, as alluded to above, we know that these conditional expectations exist as continuous (in fact, smooth) functions of $y \in M$.

2. Some large deviation results

Starting from (1.8) and (1.9), we will apply in this section results from [3] to analyze the limit behavior of the first two logarithmic derivatives of $p_T(\,\cdot\,,y)$ as $T \searrow 0$; and for this purpose, we must begin by reviewing some terminology. In particular, recall (cf. Sec. 2 of Chap. 13 in [1]) that, for a given $y \in M$, the *cut-locus* $C_m(y)$ of y is the set of $x \in M$ for which at least one of the following conditions obtains:

(i) There exists more than one minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining x to y.

(ii) There is precisely one minimizing geodesic $\gamma : [0,1] \longrightarrow M$ from x to y, and there exists along γ a non-trivial Jacobi field $t \in [0,1] \longmapsto W(t) \in$

$T_{\gamma(t)}(M)$ which vanishes at both end points; that is:

$$W(0)=\mathbf{0},\ \frac{DW}{dt}(0)\neq\mathbf{0},\ W(1)=\mathbf{0},$$
$$\text{and}\quad \frac{D^2W}{dt^2}(t)+\mathrm{Riem}\big(\dot\gamma(t),W(t)\big)\dot\gamma(t)=0.$$

Equivalently, $x\in C_m(y)$ if and only if either $\exp_x^{-1}(y)$ contains more than one element or $\exp_x^{-1}(y)=\{X_x\}$ for some $X_x\in T_x(M)$ and the map $\exp_x(X_x)_*:T_{X_x}\big(T_x(M)\big)\longrightarrow T_y(M)$ is singular. From this latter description, it is an easy matter to see that $C_m(y)$ is closed.

The relevance of these considerations to us is most easily seen after one introduces the following constructions. Namely, let $x\in M$ and $\mathfrak{e}\in\pi^{-1}(x)$ be given, and take $W^{(1)}_{\mathfrak{e},2}\big(\mathcal{O}(\mathcal{M})\big)$ to be the space of absolutely continuous curves $\mathfrak{F}_e:[0,1]\longrightarrow\mathcal{O}(\mathcal{M})$ with the properties that

$$\mathfrak{F}_\mathfrak{e}(0)=\mathfrak{e},\ \omega\big(\dot{\mathfrak{F}}_e(\,\cdot\,)\big)\in L^2\big([0,1];\mathbb{R}^d\big),\ \text{and}\ \phi\big(\dot{\mathfrak{F}}_\mathfrak{e}(\,\cdot\,)\big)=\mathbf{0}\ \text{a.e..}$$

Equivalently, if $\mathbf{H}=W^{(1)}_{\mathbf{0},2}(\mathbb{R}^d)$ is the space of absolutely continuous $\mathbb{R}^d$-valued functions $\mathbf{h}$ on $[0,1]$ with $\mathbf{h}(0)=\mathbf{0}$ and $\dot{\mathbf{h}}\in L^2\big([0,1];\mathbb{R}^d\big)$, then

$$W^{(1)}_{\mathfrak{e},2}\big(\mathcal{O}(\mathcal{M})\big)=\big\{\mathfrak{F}_\mathfrak{e}(\,\cdot\,,\mathbf{h}):\ \mathbf{h}\in\mathbf{H}\big\},$$

where $t\in[0,1]\longmapsto\mathfrak{F}_\mathfrak{e}(t,\mathbf{h})\in\mathcal{O}(\mathcal{M})$ is determined by[2]

$$\dot{\mathfrak{F}}_\mathfrak{e}(t,\mathbf{h})=\mathcal{E}\big(\dot{\mathbf{h}}(t)\big)_{\mathfrak{F}_\mathfrak{e}(t,\mathbf{h})}\quad\text{with}\ \mathfrak{F}_\mathfrak{e}(0,\mathbf{h})=\mathfrak{e}.$$

In fact, we can give $W^{(1)}_{\mathfrak{e},2}\big(\mathcal{O}(\mathcal{M})\big)$ the structure of a Polish space by declaring $\mathbf{h}\in\mathbf{H}\longmapsto\mathfrak{F}_\mathfrak{e}(\,\cdot\,,\mathbf{h})\in W^{(1)}_{\mathfrak{e},2}\big(\mathcal{O}(\mathcal{M})\big)$ to be an isometry. Notice that

$$\text{(2.1)}\qquad \mathbf{h}_n\xrightarrow{\ \mathrm{w}\ }\mathbf{h}\ \text{in}\ \mathbf{H}\implies\mathfrak{F}_\mathfrak{e}(\,\cdot\,,\mathbf{h}_n)\longrightarrow\mathfrak{F}_\mathfrak{e}(\,\cdot\,,\mathbf{h})\ \text{in}\ C\big([0,1];\mathcal{O}(\mathcal{M})\big).$$

In particular, for each $y\in M$,

$$\text{(2.2)}\qquad \mathbf{H}(\mathfrak{e},y)\equiv\big\{\mathbf{g}\in\mathbf{H}:\ \pi\circ\mathfrak{F}_\mathfrak{e}(1,\mathbf{g})=y\big\}$$

is closed in the weak topology.

[2]For each $\mathbf{v}\in\mathbb{R}^d$, $\mathcal{E}(\mathbf{v})$ is the vector field on $\mathcal{O}(\mathcal{M})$ such that, for each $\mathfrak{e}\in\mathcal{O}(\mathcal{M})$, $\mathcal{E}(\mathbf{v})_\mathfrak{e}$ is the horizontal lift of $\mathfrak{e}\mathbf{v}\in T_{\pi(\mathfrak{e})}(M)$ to $\mathfrak{e}$.

Next, given $(\mathbf{g},\mathbf{h}) \in \mathbf{H}^2$, there is (cf. Theorem 2.5 in [2]) a unique absolutely continuous path $s \in \mathbb{R} \longmapsto [\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(\cdot,\mathbf{g})](s) \in W^{(1)}_{\mathfrak{e},2}(\mathcal{O}(\mathcal{M}))$ with the properties that[3]

$$(2.3) \qquad [\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})](0) = \mathfrak{F}_{\mathfrak{e}}(t,\mathbf{g}) \quad \text{and} \quad \omega\big([\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})]'(s)\big) = \mathbf{h}(t)$$

for all $t \in [0,1]$ and $s \in \mathbb{R}$. In fact[4], if, for $\mathbf{g} \in \mathbf{H}$,

$$s \in \mathbb{R} \longrightarrow [\Theta_{\mathfrak{e},\mathbf{h}}(\cdot,\mathbf{g})](s) \in W^{(1)}_2(\mathbb{R}^d)$$

is determined by the equation

$$(2.4) \qquad [\dot{\Theta}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})]'(s) = \dot{\mathbf{h}}(t) - \phi\big([\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})]'(s)\big)[\Theta_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})](s)$$

with $\Theta_{\mathfrak{e},\mathbf{h}}(0,\mathbf{g}) = \mathbf{0}$ and $[\Theta_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})](0) = \dot{\mathbf{g}}(t)$, then

$$(2.5) \qquad [\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})](s) = \mathfrak{F}_{\mathfrak{e}}\big(t, [\Theta_{\mathfrak{e},\mathbf{h}}(\cdot,\mathbf{g})](s)\big).$$

Finally, set

$$(2.6) \qquad \mathbf{H}_0 = \{\mathbf{h} \in \mathbf{H} : \mathbf{h}(1) = \mathbf{0}\},$$

and notice that, for any $y \in M$,

$$(2.7) \quad (\mathbf{g},\mathbf{h}) \in \mathbf{H}(\mathfrak{e},y) \times \mathbf{H}_0 \implies [\Theta_{\mathfrak{e},\mathbf{h}}(\cdot,\mathbf{g})](s) \in \mathbf{H}(\mathfrak{e},y) \quad \text{for all } s \in \mathbb{R}.$$

Lemma 2.8. *Given $(\mathfrak{e},y) \in \mathcal{O}(\mathcal{M}) \times M$ and $\mathbf{g} \in \mathbf{H}(\mathfrak{e},y)$, define $\Psi_{\mathfrak{e},\mathbf{g}} : \mathbf{H} \longrightarrow \mathbf{H}$ so that (cf. (2.4) and (2.5)) $\Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{h}) = [\Theta_{\mathfrak{e},\mathbf{h}}(\cdot,\mathbf{g})](1)$. Then $\Psi_{\mathfrak{e},\mathbf{g}}$ maps $\mathbf{H}_0$ into $\mathbf{H}(\mathfrak{e},y)$; and, for each $r_0 > 0$, there exist $(r_1,r_2) \in (0,\infty)^2$ such that $\Psi_{\mathfrak{e},\mathbf{g}} \restriction \mathbf{H}_0 \cap B_{\mathbf{H}}(\mathbf{0},r_1)$[5] is a diffeomorphism onto a neighborhood of $\mathbf{H}(\mathfrak{e},y) \cap B_{\mathbf{H}}(\mathbf{g},r_2)$ whenever $\|\mathbf{g}\|_{\mathbf{H}} \le r_0$.*

Proof. First observe that

$$[D_{\mathbf{h}}\Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{0})](t) \equiv \frac{d}{ds}[\Psi_{\mathfrak{e},\mathbf{g}}(s\mathbf{h})](t)\Big|_{s=0} = [\Theta_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})]'(0).$$

Thus, by (2.4) and the second structural equation,

$$\begin{aligned}\frac{d}{dt}[D_{\mathbf{h}}\Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{0})](t) &= \dot{\mathbf{h}}(t) - \phi\big([\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\mathbf{g})]'(0)\big)\dot{\mathbf{g}}(t) \\ &= \dot{\mathbf{h}}(t) - \int_0^t \Phi\big(\dot{\mathbf{g}}(\tau),\mathbf{h}(\tau)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\mathbf{g})}\dot{\mathbf{g}}(t)\,d\tau,\end{aligned}$$

[3]In the following, and hereafter, we use prime to indicate differentiation with respect to s.

[4]See Lemma 2.5 in [2] for more details.

[5]We use $B_E(a,r)$ to denote the ball of radius r around a in the metric space E.

and therefore the homomorphism

$$\mathbf{h} \in \mathbf{H} \longmapsto D\Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{0})\mathbf{h} \equiv D_{\mathbf{h}}\Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{0}) \in \mathbf{H}$$

admits an inverse whose bound can be made to depend on $\|\mathbf{g}\|_{\mathbf{H}}$ alone. In particular, by the Implicit Function Theorem, for each $r_0 > 0$, there is an $r_1 > 0$ and an open neighborhood U of $\mathbf{0}$ in $\mathbf{H}$ such that $\Psi_{\mathfrak{e},\mathbf{g}}$ maps $B_{\mathbf{H}}(\mathbf{0}, r_1)$ diffeomorphically onto a neighborhood of $\mathbf{g}+U$ whenever $\|\mathbf{g}\|_{\mathbf{H}} \le r_0$. Furthermore, since, by (2.7), we already know that $\Psi_{\mathfrak{e},\mathbf{g}}$ takes $\mathbf{H}_0$ into $\mathbf{H}(\mathfrak{e},\mathbf{g})$, we will have completed the proof of the first part once we show that there is an $r_2 > 0$ such that $\Psi^{-1}_{\mathfrak{e},\mathbf{g}}\big(\mathbf{H}(\mathfrak{e},y)\cap B_{\mathbf{H}}(\mathbf{g},r_2)\big) \subseteq \mathbf{H}_0$ for all $\|\mathbf{g}\|_{\mathbf{H}} \le r_0$. To this end, we apply the form of the Implicit Function Theorem given in Theorem A.2 of [3] to the map $\mathbf{h} \in \mathbf{H} \longmapsto F_{\mathfrak{e}}(\mathbf{h}) \equiv \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{h}) \in M$. Indeed, because the Jacobian $DF_{\mathfrak{e}}(\mathbf{g}) : \mathbf{H} \longrightarrow T_y(M)$ has full rank, that theorem states that there is a smooth map $\Xi_{\mathfrak{e},\mathbf{g}} : \ker\big(DF(\mathbf{g})\big) \longrightarrow \ker\big(DF_{\mathfrak{e}}(\mathbf{g})\big)^{\perp}$ and positive r_2 and δ such that

$$\begin{gathered}\mathbf{h} \in B_{\mathbf{H}}(\mathbf{0}, r_2) \cap \ker\big(DF_{\mathfrak{e}}(\mathbf{g})\big) \implies \\ \Xi_{\mathfrak{e},\mathbf{g}}(\mathbf{h}) \text{ is the unique element of } B_{\mathbf{H}}(\mathbf{0},\delta) \cap \ker\big(DF_{\mathfrak{e}}(\mathbf{g})\big)^{\perp} \\ \text{with } F_{\mathfrak{e}}\big(\mathbf{g}+\mathbf{h}+\Xi_{\mathfrak{e},\mathbf{g}}(\mathbf{h})\big) = y\end{gathered}$$

whenever $\|\mathbf{g}\|_{\mathbf{H}} \le r_0$. Further, by making δ smaller if necessary, we may assume that $\Psi_{\mathfrak{e},\mathbf{g}}$ maps a neighborhood of $\mathbf{0}$ diffeomorphically onto a neighborhood of $\{\mathbf{g}+\mathbf{h}+\Xi_{\mathfrak{e},\mathbf{g}}(\mathbf{h}) : \mathbf{h} \in B_{\mathbf{H}}(\mathbf{0},r)\}$ and that the size of these neighborhoods does not depend on $\mathfrak{e} \in \mathcal{O}(\mathcal{M})$ or $\mathbf{g} \in \overline{B_{\mathbf{H}}(\mathbf{0},r_0)}$. Now let $(\mathfrak{e},\mathbf{g}) \in \mathcal{O}(\mathcal{M}) \times B_{\mathbf{H}}(\mathbf{0},r_0)$ and $\mathbf{f} \in B_{\mathbf{H}}(\mathbf{g},r_2) \cap H(\mathfrak{e},y)$ be given, set $\mathbf{k}$ equal the orthogonal projection of $\mathbf{f}-\mathbf{g}$ onto $\ker\big(DF_{\mathfrak{e}}(\mathbf{g})\big)$, and define $\mathbf{f}_s = \mathbf{g} + s\mathbf{k} + \Xi(s\mathbf{k})$ for $s \in [0,1]$. Clearly, $\mathrm{f}_0 = \mathbf{g}$, $\mathrm{f}_1 = \mathbf{f}$, and $\mathbf{f}_s \in \mathbf{H}(\mathfrak{e},y)$ for all $s \in [0,1]$. Thus, if $\mathbf{h}_s = \Psi^{-1}_{\mathfrak{e},\mathbf{g}}(\mathbf{f}_s)$, then

$$\mathbf{f} = \Psi_{\mathfrak{e},\mathbf{g}}(\mathbf{h}_1),\ \mathbf{h}_0 \in \mathbf{H}_0,\ \text{and } \pi \circ \big[\mathfrak{F}_{\mathfrak{e},\mathbf{h}_s}(1,\mathbf{g})\big](1) = y \text{ for each } s \in [0,1].$$

In particular,

$$\mathbf{h}_0(1) = \mathbf{0} \quad \text{and} \quad \mathbf{0} = \omega\left(\frac{d}{ds}\big[\mathfrak{F}_{\mathfrak{e},\mathbf{h}_s}(1,\mathbf{g})\big](1)\right) = \frac{d}{ds}\mathbf{h}_s(1),$$

and so $\Psi^{-1}_{\mathfrak{e},\mathbf{g}}(\mathbf{f}) = \mathbf{h}_1 \in \mathbf{H}_0$. q.e.d.

Lemma 2.9. *There is a unique minimal geodesic $\gamma_x : [0,1] \longrightarrow M$ from x to y if and only if, for each $\mathfrak{e} \in \pi^{-1}(x)$, there is a unique $\ell_{\mathfrak{e}} \in \mathbf{H}(\mathfrak{e},y)$ with $\|\ell_{\mathfrak{e}}\|_{\mathbf{H}} = \mathrm{dist}(x,y)$, in which case*

$$(2.10) \qquad \ell_{\mathfrak{e}}(t) = t\theta_{\mathfrak{e}},\ t \in [0,1], \quad \textit{where } \theta_{\mathfrak{e}} = \mathfrak{e}^{-1}\dot{\gamma}_x(0).$$

In fact, if there is only one such minimal geodesic γ_x and if $\theta_{\mathfrak{e}}$ and $\ell_{\mathfrak{e}}$ are defined accordingly, as in (2.10), then $x \notin C_m(y)$ if and only if there exists an $\epsilon_x > 0$ for which the symmetric quadratic form given by

$$[E''(\mathfrak{e},y)](\mathbf{g},\mathbf{h}) \equiv (\mathbf{g},\mathbf{h})_{\mathbf{H}} + \int_0^1 \Big(\Phi\big(\theta_{\mathfrak{e}},\mathbf{g}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}},\dot{\mathbf{h}}(t)\Big)_{\mathbb{R}^d}\,dt$$

satisfies

$$[E''(\mathfrak{e},y)](\mathbf{h},\mathbf{h}) \geq \epsilon_x \|\mathbf{h}\|^2_{\mathbf{H}}, \quad \mathbf{h}\in\mathbf{H}_0. \tag{2.11}$$

Finally, the ϵ_x in (2.11) can be chosen to be uniformly positive on compact subsets of $M \setminus C_m(y)$.

Proof. Observe that because

$$\operatorname{dist}(x,y)^2 = \min\{\|\mathbf{g}\|^2_{\mathbf{H}} : \mathbf{g}\in\mathbf{H}(\mathfrak{e},y)\}$$

and the minimum is achieved at $\ell_{\mathfrak{e}} \in \mathbf{H}(\mathfrak{e},y)$ only if $\ell_{\mathfrak{e}}$ is linear, the first assertion follows from the fact that $\pi\circ\mathfrak{F}_{\mathfrak{e}}(\,\cdot\,,\mathbf{g})$ is a geodesic if and only if $\mathbf{g}$ is linear.

To prove the second assertion, assume that there is only one minimal geodesic γ_x from x to y on $[0,1]$, and define $\theta_{\mathfrak{e}}$ and $\ell_{\mathfrak{e}}$ as in (2.10). By (2.7), (2.5), and (2.3), we know that, for all $\mathbf{h}\in\mathbf{H}_0$:

$$\frac{d}{ds}\int_0^1 \big|\omega\big([\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})](s)\big)\big|^2\,dt = 2\int_0^1 \Big([\dot{\Theta}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})](s),\dot{\mathbf{h}}(t)\Big)_{\mathbb{R}^d}\,dt \tag{2.12}$$

vanishes at $s=0$ and that

$$\begin{aligned}0 \leq & \frac{d^2}{ds^2}\int_0^1 \big|\omega\big([\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})](s)\big|^2\,dt\Big|_{s=0}\\ =&2\int_0^1 \Big(\dot{\mathbf{h}}(t) - \phi\big([\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})]'(0)\big)\theta_{\mathfrak{e}},\dot{\mathbf{h}}(t)\Big)_{\mathbb{R}^d}\,dt,\end{aligned}$$

which, after integration by parts and an application of the second structural equation, means that

$$\begin{gathered}[E''(\mathfrak{e},y)](\mathbf{h},\mathbf{h}) = \frac{1}{2}\frac{d^2}{ds^2}\int_0^1 \big|\omega\big([\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})](s)\big|^2\,dt\Big|_{s=0} \geq 0,\\ \mathbf{h}\in\mathbf{H}_0.\end{gathered} \tag{2.13}$$

In particular, (2.13) implies that, for any $\mathbf{h}\in\mathbf{H}_0$,

$$[E''(\mathfrak{e},y)](\mathbf{h},\mathbf{h}) = 0 \iff [E''(\mathfrak{e},y)](\mathbf{h},\mathbf{g}) = 0 \text{ for all } \mathbf{g}\in\mathbf{H}_0,$$

which, after integration by parts and elementary analysis, leads to

$$\begin{aligned}&\big[E''(\mathfrak{e},y)\big](\mathbf{h},\mathbf{h})=0 \iff \mathbf{h}\in C^2\big([0,1];\mathbb{R}^d\big)\\ &\text{and } \ddot{\mathbf{h}}(t)=\Phi\big(\theta_{\mathfrak{e}},\mathbf{h}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}},\ t\in[0,1].\end{aligned} \tag{2.14}$$

To complete the proof from here, observe that $t\in[0,1]\longmapsto Y(t)\in T_{\gamma(t)}(M)$ is a Jacobi field along γ if and only if $\mathbf{h}=\mathfrak{F}_{\mathfrak{e}}(\,\cdot\,,\ell_{\mathfrak{e}})^{-1}Y$ satisfies the equation on the right-hand side of (2.14). In particular, if (2.11) holds for some $\epsilon>0$, then there is no Jacobi field along γ which vanishes at both ends. Conversely, suppose no such Jacobi field exists. Then

$$\mathbf{h}\in\mathbf{H}_0\setminus\{\mathbf{0}\}\implies\big[E''(\mathfrak{e},y)\big](\mathbf{h},\mathbf{h})>0. \tag{2.15}$$

Finally, suppose that K is a compact subset of $\pi^{-1}\big(M\setminus C_m(y)\big)$ and that there were sequences $\{\mathfrak{e}_n\}_1^\infty\subseteq K$ and $\{\mathbf{h}_n\}_1^\infty\subseteq\mathbf{H}_0$ with the properties that

$$\|\mathbf{h}_n\|_{\mathbf{H}}=1 \quad\text{and}\quad \big[E''(\mathfrak{e}_n,y)\big](\mathbf{h}_n,\mathbf{h}_n)\longrightarrow 0.$$

Then, without loss in generality, we will assume: $\mathfrak{e}_n\longrightarrow\mathfrak{e}\in\pi^{-1}\big(M\setminus C_m(y)\big)$ and $\mathbf{h}_n\xrightarrow{\text{w}}\mathbf{h}\in\mathbf{H}_0$.

Note (cf. (2.1)) that

$$\begin{aligned}&\|\mathbf{h}\|_{\mathbf{H}}\le\underline{\lim}_{n\to\infty}\|\mathbf{h}_n\|_{\mathbf{H}}\quad\text{and}\\ &\int_0^1\Big(\Phi\big(\theta_{\mathfrak{e}_n},\mathbf{h}_n(t)\big)_{\mathfrak{F}_{\mathfrak{e}_n}(t,\ell_{\mathfrak{e}_n})}\theta_{\mathfrak{e}_n},\mathbf{h}_n(t)\Big)_{\mathbb{R}^d}dt\\ &\qquad\longrightarrow\int_0^1\Big(\Phi\big(\theta_{\mathfrak{e}},\mathbf{h}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}},\mathbf{h}(t)\Big)_{\mathbb{R}^d}dt.\end{aligned}$$

In particular,

$$0\le\big[E''(\mathfrak{e},y)\big](\mathbf{h},\mathbf{h})\le\underline{\lim}_{n\to\infty}\big[E''(\mathfrak{e}_n,y)\big](\mathbf{h}_n,\mathbf{h}_n)=0,$$

and therefore $\mathbf{h}=\mathbf{0}$. But this means that

$$0=\|\mathbf{h}\|_{\mathbf{H}}^2=\big[E''(\mathfrak{e},y)\big](\mathbf{h},\mathbf{h})=\lim_{n\to\infty}\big[E''(\mathfrak{e}_n,y)\big](\mathbf{h}_n,\mathbf{h}_n)=\lim_{n\to\infty}\|\mathbf{h}_n\|_{\mathbf{H}}^2=1,$$

which is impossible. Thus there must exist an $\epsilon>0$ for which (2.14) obtains for all $x\in K$. q.e.d.

Lemma 2.16. *Let $x\notin C_m(y)$ and $\mathfrak{e}\in\pi^{-1}(x)$ be given, and define $\gamma_x\in C^\infty\big([0,1];M\big)$, $\theta_{\mathfrak{e}}\in\mathbb{R}^d$, and $\ell_{\mathfrak{e}}\in\mathbf{H}$ as in Lemma 2.9. Then, for each $\mathbf{v}\in\mathbb{R}^d$ there is a unique $\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}\in C^2\big([0,1];\mathbb{R}^d\big)$ such that*

$$\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(0)=\mathbf{v},\ \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(1)=\mathbf{0},\ \text{and } \ddot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t)=\Phi\big(\theta_{\mathfrak{e}},\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}}. \tag{2.17}$$

In fact, $(\mathfrak{e},\mathbf{v}) \in \pi^{-1}(M\backslash C_m(y))\times\mathbb{R}^d \longmapsto \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}} \in C^2([0,1];\mathbb{R}^d)$ *is continuous. Finally,*

$$\begin{aligned}(2.18)\qquad \boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}} &= \mathbf{v} - \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}} \\ &\implies \frac{d}{ds}\int_0^1 \Big(\omega\big([\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})](s)\big), \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t)\Big)_{\mathbb{R}^d}\,dt\Big|_{s=0} = 0 \\ &\qquad\qquad \textit{for all } \mathbf{h}\in\mathbf{H}_0.\end{aligned}$$

Proof. To prove that $\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}$ exists and is a continuous function, set $\sigma_{\mathfrak{e},\mathbf{v}}(s) = \exp_{\pi(\mathfrak{e})}(s\mathfrak{e}\mathbf{v})$ for $\mathfrak{e}\in\pi^{-1}(M\setminus C_m(y))$ and $s\in\mathbb{R}$. Next, let $r>0$ and a compact subset K of $\pi^{-1}(M\setminus C_m(y))$ be given, and choose $\delta>0$ so that $\sigma_{\mathfrak{e},\mathbf{v}}(s)\notin C_m(y)$ for $\mathfrak{e}\in K$, $|\mathbf{v}|\le r$, and $|s|<\delta$. Now define

$$\Sigma_{\mathfrak{e},\mathbf{v}}(s) = \exp^{-1}_{\sigma_{\mathfrak{e},\mathbf{v}}(s)}(y) \quad\text{and}\quad [\Gamma_{\mathfrak{e},\mathbf{v}}(t)](s) = \exp_{\sigma_{\mathfrak{e},\mathbf{v}}(s)}\big(t\Sigma_{\mathfrak{e},\mathbf{v}}(s)\big),\quad t\in[0,1]$$

for $\mathfrak{e}\in K$, $|\mathbf{v}|\le r$, and $|s|<\delta$. Then, as is well-known,

$$t\in[0,1]\longmapsto Y_{\mathfrak{e},\mathbf{v}}(t)\equiv[\Gamma_{\mathfrak{e},\mathbf{v}}(t)]'(0)\in T_{\gamma_{\pi(\mathfrak{e})}(t)}(M)$$

is a Jacobi field along $\gamma_{\pi(\mathfrak{e})}$, and, by construction, $Y_{\mathfrak{e},\mathbf{v}}(0)=\mathfrak{e}\mathbf{v}$ while $Y(1)=\mathbf{0}$. Hence, we can take $\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(t) = \mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})^{-1}Y_{\mathfrak{e},\mathbf{v}}(t)$, and clearly $(\mathfrak{e},\mathbf{v})\in K\times B_{\mathbb{R}^d}(\mathbf{0},r)\longmapsto\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}\in C^2([0,1];\mathbb{R}^d)$ is continuous. Moreover, to prove uniqueness, simply observe that if $\Delta(t)$ is the difference of two solutions and $Z(t)=\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})\Delta(t)$, then Z is a Jacobi field along γ_x which vanishes at both ends.

Turning to (2.18), observe (cf. (2.4)) that

$$\begin{aligned}\frac{d}{ds}&\int_0^1\Big(\omega\big([\dot{\mathfrak{F}}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})](s),\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t)\Big)_{\mathbb{R}^d}\,dt\Big|_{s=0}\\ &=\int_0^1\Big(\dot{\mathbf{h}}(t)-\phi\big([\mathfrak{F}_{\mathfrak{e},\mathbf{h}}(t,\ell_{\mathfrak{e}})]'(0)\theta_{\mathfrak{e}},\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t)\Big)_{\mathbb{R}^d}\,dt\\ &=-\int_0^1\big(\ddot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t),\mathbf{h}(t)\big)_{\mathbb{R}^d}\,dt-\int_0^1\Big(\Phi\big(\theta_{\mathfrak{e}},\mathbf{v}-\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}},\mathbf{h}(t)\Big)_{\mathbb{R}^d}\,dt\\ &=\int_0^1\Big(\ddot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t)-\Phi\big(\theta_{\mathfrak{e}},\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta,\mathbf{h}(t)\Big)_{\mathbb{R}^d}\,dt=0,\end{aligned}$$

where we have used integration by parts, the second structural equation, and the symmetry

$$\big(\Phi(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\boldsymbol{\xi}_3,\boldsymbol{\xi}_4\big)_{\mathbb{R}^d}=\big(\Phi(\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\big)_{\mathbb{R}^d}.\quad\textit{q.e.d.}$$

Theorem 2.19. *For* $\mathfrak{e} \in \pi^{-1}(M \setminus C_m(y))$, *define* $\theta_{\mathfrak{e}} \in \mathbb{R}^d$ *and* $\ell_{\mathfrak{e}} \in \mathbf{H}$ *accordingly, as in (2.10), and set (cf. (2.18) and (2.2))*

$$(2.20)\quad [E_{\mathbf{v}}(\mathfrak{e}, y)](\mathbf{g}) = \frac{\|\mathbf{g}\|_{\mathbf{H}}^2}{2} - (\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}, \mathbf{g})_{\mathbf{H}} \quad \text{for } \mathbf{g} \in \mathbf{H}(\mathfrak{e}, y) \text{ and } \mathbf{v} \in \mathbb{R}^d.$$

Then, for each compact subset K *of* $\pi^{-1}(M \setminus C_m(y))$, *there is a* $\delta > 0$ *with the properties that*

$$(2.21)\quad \begin{aligned} &(\mathfrak{e}, \mathbf{v}) \in K \times B_{\mathbb{R}^d}(\mathbf{0}, \delta) \implies \ell_{\mathfrak{e}} \text{ is the only } \mathbf{g} \in \mathbf{H}(\mathfrak{e}, y) \\ &\text{at which } E_{\mathbf{v}}(\mathfrak{e}, y) \text{ achieves its minimum value and} \\ &\frac{d^2}{ds^2}[E_{\mathbf{v}}(\mathfrak{e}, y)]\left(\int_0^{\cdot} \omega\big([\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(\tau, \ell_{\mathfrak{e}})](s)\big)\, d\tau\right)\bigg|_{s=0} > 0 \\ &\qquad\qquad\qquad\qquad \text{for } \mathbf{h} \in \mathbf{H}_0 \setminus \{\mathbf{0}\}. \end{aligned}$$

Proof. Because of the equality in (2.13) and the estimate in (2.11), we know that there is an $\epsilon > 0$ and a $\delta_1 > 0$ for which

$$\frac{d^2}{ds^2}[E_{\mathbf{v}}(\mathfrak{e}, y)]\left(\int_0^{\cdot} \omega[\dot{\mathfrak{F}}_{\mathfrak{e},\mathbf{h}}(\tau, \ell_{\mathfrak{e}})](s)\, ds\right)\bigg|_{s=0} \geq \frac{\epsilon}{2}\|\mathbf{h}\|_{\mathbf{H}}^2,$$

whenever $(\mathfrak{e}, \mathbf{v}, \mathbf{h}) \in K \times B_{\mathbb{R}^d}(\mathbf{0}, \delta_1) \times \mathbf{H}_0$. Moreover, by Lemma 2.8, one knows that there exist positive r_1 and r_2 such that

$$(2.22)\quad \begin{aligned} &\mathbf{h} \in \mathbf{H}_0 \cap B_{\mathbf{H}}(\mathbf{0}, r_1) \longmapsto [\Theta_{\mathfrak{e},\mathbf{h}}(\,\cdot\,, \ell_{\mathfrak{e}})](1) \in \mathbf{H}(\mathfrak{e}, y) \\ &\text{is diffeomorphic onto a neighborhood of } \mathbf{H}(\mathfrak{e}, y) \cap B_{\mathbf{H}}(\mathbf{0}, r_2) \end{aligned}$$

for every $\mathfrak{e} \in \mathcal{O}(\mathcal{M})$. Hence, by (2.18) and Taylor's Theorem, for some $r > 0$,

$$(2.23)\quad \mathbf{g} \in \mathbf{H}(\mathfrak{e}, y) \cap \big(B_{\mathbf{H}}(\ell_{\mathfrak{e}}, r) \setminus \{\ell_{\mathfrak{e}}\}\big) \implies [E_{\mathbf{v}}(\mathfrak{e}, y)](\mathbf{g}) > [E_{\mathbf{v}}(\mathfrak{e}, y)](\ell_{\mathfrak{e}})$$

whenever $(\mathfrak{e}, \mathbf{v}) \in K \times B_{\mathbb{R}^d}(\mathbf{0}, \delta_1)$.

Starting from (2.23), one can now show that there is an $\alpha > 0$ with the property that

$$(2.24)\quad \begin{aligned} &\mathfrak{e} \in K,\ \mathbf{g} \in \mathbf{H}(\mathfrak{e}, y) \setminus \{\ell_{\mathfrak{e}}\}, \\ &\text{and } [E_{\mathbf{v}}(\mathfrak{e}, y)](\mathbf{g}) \geq [E_{\mathbf{v}}(\mathfrak{e}, y)](\ell_{\mathfrak{e}}) \implies \|\mathbf{g}\|_{\mathbf{H}} \geq \|\ell_{\mathfrak{e}}\|_{\mathbf{H}} + \alpha, \end{aligned}$$

whenever $|\mathbf{v}| < \delta_1$. In fact, if this were not the case, then we could use (2.23) to find a $\{\mathfrak{e}_n\}_1^\infty \subseteq K$ and $\{\mathbf{g}_n\}_1^\infty \subseteq \mathbf{H}(\mathfrak{e}_n, y) \setminus B_{\mathbf{H}}(\ell_{\mathfrak{e}_n}, r)$ such that $\mathfrak{e}_n \longrightarrow \mathfrak{e} \in K$, (cf. (2.2)) $\mathbf{g}_n \overset{\mathrm{w}}{\longrightarrow} \mathbf{g} \in \mathbf{H}(\mathfrak{e}, y)$, and $\|\mathbf{g}_n\|_{\mathbf{H}} \longrightarrow \|\ell_{\mathfrak{e}}\|_{\mathbf{H}}$. Since this means that

$$\|\ell_{\mathfrak{e}}\|_{\mathbf{H}} \leq \|\mathbf{g}\|_{\mathbf{H}} \leq \underline{\lim}_{n\to\infty}\|\mathbf{g}_n\|_{\mathbf{H}} = \|\ell_{\mathfrak{e}}\|_{\mathbf{H}},$$

$\mathbf{g}_n \longrightarrow \mathbf{g}$, $\|\mathbf{g}\|_{\mathbf{H}} = \|\ell_{\mathfrak{e}}\|_{\mathbf{H}}$, and therefore $\mathbf{g} = \ell_{\mathfrak{e}}$. On the other hand, since, $\|\mathbf{g}_n - \ell_{\mathfrak{e}}\|_{\mathbf{H}} \geq r$, this is impossible.

Finally, to complete the proof, suppose that $(\mathfrak{e}, \mathbf{v}) \in K \times B_{\mathbb{R}^d}(\mathbf{0}, \delta_1)$ and that $\mathbf{g} \in \mathbf{H}(\mathfrak{e}, y)$ satisfies $\big[E_{\mathbf{v}}(\mathfrak{e}, y)\big](\mathbf{g}) \geq \big[E_{\mathbf{v}}(\mathfrak{e}, y)\big](\ell_{\mathfrak{e}})$. Then, by Schwarz's inequality and elementary manipulation of quadratics, $\|\mathbf{g}\|_{\mathbf{H}} \leq \|\ell_{\mathfrak{e}}\|_{\mathbf{H}} + 6\|\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}\|_{\mathbf{H}}$. Hence (cf. Lemma 2.16) we can choose $0 < \delta < \delta_1$ so that (2.24) guarantees that we are done. q.e.d.

We are now ready to formulate the conclusions of Theorem 2.1 and Theorem 3.12 in [4] so that they may be easily applied to the expressions in (1.8) and (1.9). In the following statement, $g : (0,1] \times \mathfrak{W} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ will be a function of one of the following forms:

$$g(T, \mathbf{w}, \mathfrak{e}) = \begin{cases} \displaystyle\int_0^1 \varphi\big(t, \mathbf{w}, \mathfrak{F}_{\mathfrak{e}}(t, \mathbf{w}), \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w})\boldsymbol{\alpha}(t)\big)\, dt, \\ \displaystyle\int_0^1 \Big(\boldsymbol{\psi}\big(\mathfrak{F}_{\mathfrak{e}}(t, \mathbf{w}), \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w})\boldsymbol{\alpha}(t)\big), \circ d\mathbf{w}(t)\Big)_{\mathbb{R}^d}, \\ \displaystyle\int_0^1 \Big(\int_0^t \boldsymbol{\Psi}\big(\mathfrak{F}_{\mathfrak{e}}(\tau, \mathbf{w}), \mathbf{A}_{\mathfrak{e},T}(\tau, \mathbf{w})\alpha(\tau); \mathfrak{F}_{\mathfrak{e}}(t, \mathbf{w}), \\ \qquad\qquad \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w})\boldsymbol{\beta}(t)\big) \circ d\mathbf{w}(\tau), \circ d\mathbf{w}(t)\Big)_{\mathbb{R}^d}, \end{cases}$$

where[6]

$$\varphi \in C^\infty_{\nearrow}\Big([0,1] \times \mathfrak{W} \times \mathcal{O}(\mathcal{M}) \times \mathbb{R}^d; \mathbb{R}\Big), \quad \boldsymbol{\psi} \in C^\infty_{\nearrow}\big(\mathcal{O}(\mathcal{M}) \times \mathbb{R}^d; \mathbb{R}^d\big), \\ \text{and } \boldsymbol{\Psi} \in C^\infty_{\nearrow}\big(\mathcal{O}(\mathcal{M}) \times \mathbb{R}^d \times \mathcal{O}(\mathcal{M}) \times \mathbb{R}^d; \mathrm{Hom}(\mathbb{R}^d; \mathbb{R}^d)\big),$$

and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are smooth $\mathbb{R}^d$-valued paths on $[0,1]$. Finally, for $\ell \in \mathbf{H}$, we take, respectively,

$$g(0, \ell, \mathfrak{e}) = \begin{cases} \displaystyle\int_0^1 \varphi\big(t, \ell, \mathfrak{F}_{\mathfrak{e}}(t, \ell), \boldsymbol{\alpha}(t)\big)\, dt, \\ \displaystyle\int_0^1 \Big(\boldsymbol{\psi}\big(\mathfrak{F}_{\mathfrak{e}}(t, \ell), \boldsymbol{\alpha}(t)\big), \dot{\ell}(t)\Big)_{\mathbb{R}^d} dt, \\ \displaystyle\int_0^1 \Big(\boldsymbol{\Psi}\big(\mathfrak{F}_{\mathfrak{e}}(\tau, \ell), \boldsymbol{\alpha}(\tau); \\ \qquad\qquad \mathfrak{F}_{\mathfrak{e}}(t, \ell), \boldsymbol{\beta}(t)\big)\dot{\ell}(\tau), \dot{\ell}(t)\Big)_{\mathbb{R}^d} dt. \end{cases}$$

Obviously, $g(0, \ell, \mathfrak{e}) = \lim_{T \searrow 0} g(T, \ell, \mathfrak{e})$ when one adopts the convention that Stratonovich calculus reverts to ordinary calculus when dealing with absolutely continuous paths.

[6] $f \in C^\infty_{\nearrow}$ if $f \in C^\infty$ and all its derivatives have tempered growth.

Theorem 2.25. *Define* $\theta_{\mathfrak{e}} \in \mathbb{R}^d$, $\ell_{\mathfrak{e}} \in \mathbf{H}$, *and* $\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}} \in \mathbf{H}$, $\mathbf{v} \in \mathbb{R}^d$, *for* $\mathfrak{e} \in \pi^{-1}\big(M \setminus C_m(y)\big)$ *as in (2.10) and (2.18), and set*

$$f_{\mathbf{v}}(T, \mathbf{w}, \mathfrak{e}) = \int_0^1 \big(\mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w})\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t), d\mathbf{w}(t)\big)_{\mathbb{R}^d} - \big(\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}, \ell_{\mathfrak{e}}\big)_{\mathbf{H}}. \tag{2.26}$$

Next, given a compact $K \subseteq \pi^{-1}\big(M \setminus C_m(y)\big)$, *choose* $\delta > 0$ *as in Theorem 2.19. Then, for* $\mathbf{v} \in B_{\mathbb{R}^d}(0,1)$ *and* $\alpha \in (-\delta, \delta)$,

$$\begin{aligned} \limsup_{T\searrow 0} \sup_{\mathfrak{e}\in K} \Bigg|\mathbb{E}^{\mu_T} \Bigg[g(T, \mathbf{w}, \mathfrak{e}) \exp\left(\frac{\alpha f_{\mathbf{v}}(T, \mathbf{w}, \mathfrak{e})}{T}\right) \Bigg| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1, \mathbf{w}) = y\Bigg] \\ -c(\alpha, \mathfrak{e})g(0, \ell_{\mathfrak{e}}, \mathfrak{e})\Bigg| = 0, \end{aligned} \tag{2.27}$$

where $\mathfrak{e} \in K \longmapsto c(\alpha, \mathfrak{e}) \in (0, \infty)$ *is a continuous function which is equal to* 1 *when* $\alpha = 0$.

Proof. The proof of (2.27) comes down to checking that the hypotheses of Theorem 4.21 in [3] are met. In matching the notation there with that here, one should use the following table:

there	**here**
s	T
y	$\mathfrak{e}$
θ	$\mathbf{w}$
$f(s,\theta,y)$	$\alpha f_{\mathbf{v}}(T, \mathbf{w}, \mathfrak{e})$
$F(s,\theta,y)$	$\pi \circ \mathfrak{F}_{\mathfrak{e}}(1, \mathbf{w})$
$\rho(y)$	$-\frac{1}{2}\|\ell_{\mathfrak{e}}\|^2_{\mathbf{H}}$
$g(s,\theta,y)$	$g(T,\mathbf{w}, \mathfrak{e})$.

The critical fact to be observed is that, because $|\alpha\mathbf{v}| < \delta$ and $\alpha f_{\mathbf{v}} = f_{\alpha\mathbf{v}}$, (2.21) guarantees that the conditions in (4.14) and (4.22) of [3] are satisfied. In addition, one should note that, because $\mathbf{A}_T(t, \mathbf{w})\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}(t)$ is continuously differentiable with respect to $t \in [0, 1]$, there is no problem coming from the Itô stochastic integral, which can, in fact, be replaced by a Riemann-Stieltjes integral. In particular, for each $\mathbf{v} \in \mathbb{R}^d$, one can use (1.3) to see that there is a $B(\mathbf{v}) < \infty$ for which

$$\big|f_{\mathbf{v}}(T, \mathbf{w}, \mathfrak{e})\big| \le B(\mathbf{v})\big(T\|\mathbf{w}\|_{\mathfrak{W}} + \|\mathbf{w} - \ell_{\mathfrak{e}}\|_{\mathfrak{W}}\big), \quad \mathfrak{e} \in K.$$

Knowing that Theorem 4.21 of [3] applies, we conclude that there is, for each $|\alpha| < \delta$, a continuous $\mathfrak{e} \in K \longmapsto C(\alpha, \mathfrak{e}) \in (0, \infty)$ such that

$$\lim_{T \searrow 0} T^{\frac{d}{2}} \exp\left(\frac{\|\ell_{\mathfrak{e}}\|_{\mathbf{H}}^2}{2T}\right) \mathbb{E}^{\mu_T}\left[g(T, \mathbf{w}, \mathfrak{e}) \exp\left(\frac{\alpha f_{\mathbf{v}}(T, \mathbf{w}, \mathfrak{e})}{T}\right) \delta_y\big(\pi \circ \mathfrak{F}_{\mathfrak{e}}(1, \mathbf{w})\big)\right] = C(\alpha, \mathfrak{e}) g(0, \ell_{\mathfrak{e}}, \mathfrak{e})$$

uniformly in $\mathfrak{e} \in K$ for each choice of the functions φ, ψ, and $\boldsymbol{\Psi}$ entering the definition of g. In particular, by applying this result again when $\alpha = 0$ and $g \equiv 1$ and then taking ratios to get the conditional expectation value, one arrives at (2.27). q.e.d.

Corollary 2.29. *For $x, y \in M$, let $E(x,y) = \frac{1}{2}\operatorname{dist}(x,y)^2$. Then $E(\cdot\,, y)$ is a smooth function on $M \backslash C_m(y)$. Moreover,*

$$\lim_{T \searrow 0}\big[T \nabla \log p_T(\cdot\,, y)\big](\mathfrak{e}) = -\big[\nabla E(\cdot\,, y)\big](\mathfrak{e}) \tag{2.30}$$

and

$$\lim_{T \searrow 0}\big[T \operatorname{Hess} \log p_T(\cdot\,, y)\big](\mathfrak{e}) = -\big[\operatorname{Hess} E(\cdot\,, y)\big](\mathfrak{e}) \tag{2.31}$$

uniformly on compact subsets of $M \setminus C_m(y)$.

Proof. Define $\theta_{\mathfrak{e}} \in \mathbb{R}^d$, $\ell_{\mathfrak{e}} \in \mathbf{H}$, as in (2.10), and $\mathbf{v} \in \mathbb{R}^d \longmapsto \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}} \in \mathbf{H}$, as in Lemma 2.16, for $\mathfrak{e} \in \pi^{-1}\big(M \setminus C_m(y)\big)$.

We begin by pointing out that when one translates the results in Section 2 of Chapter 9 [1] into the language of the orthogonal frame bundle, one finds that

$$\begin{aligned} \Big(\big[\nabla E(\cdot\,, y)\big](\mathfrak{e}), \mathbf{v}\Big)_{\mathbb{R}^d} &= \big[\mathcal{E}(\mathbf{v}) E(\cdot\,, y)\big](\mathfrak{e}) = -\big(\mathbf{v}, \theta_{\mathfrak{e}}\big)_{\mathbb{R}^d} \\ &= \big(\boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}, \ell_{\mathfrak{e}}\big)_{\mathbf{H}} \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} &\Big(\mathbf{v}, \big[\operatorname{Hess} E(\cdot\,, y)\big](\mathfrak{e})\mathbf{v}\Big)_{\mathbb{R}^d} \\ &\quad = \big[\mathcal{E}(\mathbf{v})^2 E(\cdot\,, y)\big](\mathfrak{e}) \\ &\quad = \int_0^1 \Big(\big|\dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t)\big|^2 - \Big(\Phi\big(\theta_{\mathfrak{e}}, \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(t,\ell_{\mathfrak{e}})}\theta_{\mathfrak{e}}, \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(t)\Big)_{\mathbb{R}^d}\Big)\, dt. \end{aligned} \tag{2.33}$$

Thus, what we have[7] to do is to show that $-T\big[\mathcal{E}(\mathbf{v}) \log p_T(\cdot\,, y)\big](\mathfrak{e})$ and $-T\big[\mathcal{E}(\mathbf{v})^2 \log p_T(\cdot\,, y)\big](\mathfrak{e})$ tend, uniformly on compacts, to the right-hand

[7]Actually, from general principles, (0.3) plus locally bounded convergence of the left hands sides already guarantees that the right-hand sides of (2.32) and (2.33) must be what they are.

sides of (2.32) and (2.33), respectively. In particular, (2.30) is now an easy consequence of (1.8), with $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}$, and (2.27) with $\alpha = 0$ and

$$g(T, \mathbf{w}, \mathfrak{e}) = \int_0^1 \Big(\mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t), d\mathbf{w}(t) \Big)_{\mathbb{R}^d}$$
$$= - \int_0^1 \Big(\mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t), \circ d\mathbf{w}(t) \Big)_{\mathbb{R}^d},$$

since there is no distinction between Itô and Stratonovich integration here and $\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}} = -\dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}$.

The verification of (2.31) is somewhat more involved and consists of several steps. First one should notice that, when $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}$, the first two terms of the right-hand side of (1.9) can be rewritten in the following ways:

$$\int_0^1 \left| \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t) \right|^2 dt = \int_0^1 \left| \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t) \right|^2 dt,$$

and

$$\int_0^1 \Big(\phi_{\mathfrak{e},T,\boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}}(t, \mathbf{w}) \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t), \circ d\mathbf{w}(t) \Big)_{\mathbb{R}^d}$$
$$= \int_0^1 \left(\int_0^t \boldsymbol{\Psi}_{\mathfrak{e},T,\mathbf{v}}(\tau, t, \mathbf{w}) \circ d\mathbf{w}(\tau), \circ d\mathbf{w}(t) \right)_{\mathbb{R}^d}$$

where

$$\boldsymbol{\Psi}_{\mathfrak{e},T,\mathbf{v}}(\tau, t, \mathbf{w})_{i,j} \equiv \Big(\Phi\big(\mathbf{e}_j, \mathbf{A}_{\mathfrak{e},T}(t, \mathbf{w}) \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(\tau,\mathbf{w})} \mathbf{e}_i, \mathbf{A}_{\mathfrak{e},T}(\tau, \mathbf{w}) \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(\tau) \Big)_{\mathbb{R}^d}$$

for $1 \le i, j \le d$. At the same time, the third term on the right of (1.9) is dominated (cf. (1.7)) by a constant times

$$T \int_0^1 |\mathbf{w}(1) - \mathbf{w}(t)|^2 \, dt.$$

Hence, by taking $\alpha = 0$ and $g(T, \mathbf{w}, \mathfrak{e})$ appropriately, (1.9), with $\boldsymbol{\eta} = \boldsymbol{\eta}_{\mathfrak{e},\mathbf{v}}$, and (2.27) lead to

$$\lim_{T \searrow 0} \left[T\big[\mathcal{E}(\mathbf{v})^2 \log p_T(\,\cdot\,, y)\big](\mathfrak{e}) - \frac{1}{T} \operatorname{Var}_{\mathfrak{e},y,\mathbf{v}}(T) \right]$$
$$= - \int_0^1 \left| \dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t) \right|^2 dt + \int_0^1 \left(\int_0^t \Big(\Phi\big(\theta_{\mathfrak{e}}, \dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t)\big)_{\mathfrak{F}_{\mathfrak{e}}(\tau,\ell_{\mathfrak{e}})} \theta_{\mathfrak{e}}, \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(\tau) \Big)_{\mathbb{R}^d} d\tau \right) dt$$
$$= - \int_0^1 \left| \dot{\boldsymbol{\xi}}_{\mathfrak{e},\mathbf{v}}(t) \right|^2 dt + \int_0^1 \Big(\Phi\big(\theta_{\mathfrak{e}}, \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(\tau)\big)_{\mathfrak{F}_{\mathfrak{e}}(\tau,\ell_{\mathfrak{e}})} \theta_{\mathfrak{e}}, \boldsymbol{\xi}_{\mathfrak{e},\mathbf{v}}(\tau) \Big)_{\mathbb{R}^d} d\tau$$
$$= - \big[\mathcal{E}(\mathbf{v})^2 E(\,\cdot\,, y)\big](\mathfrak{e}),$$

where the convergence is uniform on compact subsets of $\pi^{-1}(M \setminus C_m(y))$ and (cf. (2.26))

$$\begin{aligned}\mathrm{Var}_{\mathfrak{e},y,\mathbf{v}}(T) &\equiv \mathbb{E}^{\mu_T}\left[\left(\int_0^1 \left(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t), d\mathbf{w}(t)\right)_{\mathbb{R}^d}\right)^2 \middle| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right] \\ &\qquad - \mathbb{E}^{\mu_T}\left[\int_0^1 \left(\mathbf{A}_{\mathfrak{e},T}(t,\mathbf{w})\dot{\boldsymbol{\eta}}_{\mathfrak{e},\mathbf{v}}(t), d\mathbf{w}(t)\right)_{\mathbb{R}^d} \middle| \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right]^2 \\ &\le \mathbb{E}^{\mu_T}\left[f_{\mathbf{v}}(T,\mathbf{w},\mathfrak{e})^2 \middle| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right].\end{aligned}$$

Thus, we will be done once we show that, for each compact $K \subseteq \pi^{-1}(M \setminus C_m(y))$,

$$\lim_{T\searrow 0} \sup_{\mathfrak{e}\in K} \frac{1}{T}\mathbb{E}^{\mu_T}\left[f_{\mathbf{v}}(T,\mathbf{w},\mathfrak{e})^2 \middle| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right] = 0.$$

But, by (2.27) with $g_T \equiv 1$, we know that there is a $\delta > 0$ for which

$$\sup_{\substack{T\in(0,1]\\ \mathfrak{e}\in K}} \mathbb{E}^{\mu_T}\left[\exp\left(\frac{\alpha f_{\mathbf{v}}(T,\mathbf{w},\mathfrak{e})}{T}\right) \middle| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right] < \infty$$

for $\alpha \in (-\delta, \delta)$, which means, of course, that

$$\sup_{\substack{T\in(0,1]\\ \mathfrak{e}\in K}} \mathbb{E}^{\mu_T}\left[\left|\frac{f_{\mathbf{v}}(T,\mathbf{w})}{T}\right|^p \middle| \pi \circ \mathfrak{F}_{\mathfrak{e}}(1,\mathbf{w}) = y\right] < \infty \tag{2.34}$$

for all $p \in [1, \infty)$. q.e.d.

When $x \in C_m(y)$, we cannot, in general, say what are the limits of the left-hand sides of (2.30) and (2.31) as $T \searrow 0$. Nonetheless, there are circumstances in which we can say something; namely, set

$$M(x,y) = \left\{X_x \in T_x(M) : y = \exp_x(X_x) \text{ and } \mathrm{dist}(x,y) = |X_x|_{T_x(M)}\right\}$$

and

$$\widehat{M(x,y)} = \left\{(X_x, W_x) \in M(x,y) \times \left(T_x(M) \setminus \{0\}\right) : \frac{d}{ds}\exp_x(X_x + sW_x)\Big|_{s=0} = \mathbf{0}\right\}$$

Clearly, $x \in C_m(y)$ if and only if either $M(x,y)$ contains more than one element or $\widehat{M(x,y)} \neq \emptyset$.

Theorem 2.35. *Assume that $M(x,y)$ contains more than one element and that there exists a submanifold $\widetilde{M(x,y)} \supseteq M(x,y)$ of $T_x(M)$ with the property that*

$$(X_x, W_x) \in \widehat{M(x,y)} \implies W_x \not\perp T_{X_x}\left(\widetilde{M(x,y)}\right).$$

Further, assume that $M(x,y)$ has positive measure when $\widetilde{M(x,y)}$ is given the measure determined by the Riemannian structure which it inherits as a submanifold. If $\mathfrak{e} \in \pi^{-1}(x)$, then there exists a non-degenerate (i.e., not concentrated at a single point) Borel probability measure $\lambda_{(\mathfrak{e},y)}$ on $\mathbb{R}^d$ which is supported on $\{\theta_\mathfrak{e} \in \mathbb{R}^d : \mathfrak{e}\theta_\mathfrak{e} \in M(x,y)\}$ and for which

$$\lim_{T \searrow 0} T\big[\mathcal{E}(\mathbf{v}) \log p_T(\,\cdot\,,y)\big](\mathfrak{e}) = -\int (\mathbf{v},\theta_\mathfrak{e})_{\mathbb{R}^d}\, \lambda_{(\mathfrak{e},y)}(d\theta_\mathfrak{e}), \quad \mathbf{v} \in \mathbb{R}^d. \tag{2.36}$$

In addition, for each $\mathbf{v} \in \mathbb{R}^d$:

$$\begin{aligned}\lim_{T \searrow 0} T^2 \quad & \big[\mathcal{E}(\mathbf{v})^2 \log p_T(\,\cdot\,,y)\big](\mathfrak{e}) \\ & \int (\mathbf{v},\theta_\mathfrak{e})^2_{\mathbb{R}^d}\, \lambda_{(\mathfrak{e},y)}(d\theta_\mathfrak{e}) - \left(\int (\mathbf{v},\theta_\mathfrak{e})_{\mathbb{R}^d}\, \lambda_{(\mathfrak{e},y)}(d\theta_\mathfrak{e})\right)^2.\end{aligned} \tag{2.37}$$

In particular,

$$\lim_{T \searrow 0} T\big|\nabla \log p_T(\,\cdot\,,y)\big|(\mathfrak{e}) < \mathrm{dist}(x,y) \tag{2.38}$$

and

$$\lim_{T \searrow 0} T^2\big[\mathrm{Hess} \log p_T(\,\cdot\,,y)\big](\mathfrak{e}) \neq \mathbf{0}. \tag{2.39}$$

Proof. Given $\mathbf{v} \in \mathbf{S}^{d-1}$, take $\boldsymbol{\eta}(t) = t\mathbf{v}$, $t \in [0,1]$, in (1.8) and (1.9). After making the same sort of dictionary as we did in the proof of Theorem 2.29, one can apply Theorem 2.1 in [4] to the expressions on the right-hand sides of (1.8) and (1.9) to obtain (2.36) and (2.37). Moreover, given these, (2.38) and (2.39) are simply expressions of the non-degeneracy of $\lambda_{(\mathfrak{e},y)}$. q.e.d.

Remark 2.40. As the reader has probably guessed, higher derivative analogs of (2.30) and (2.31) hold and can be proved by the techniques used here. In fact, sufficient diligence combined with the estimate in (2.34) are all that is required.

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